Angel's preserving linear mapping.
https://www.linkedin.com/feed/update/urn:li:activity:6949923019809058816?utm_source=linkedin_share&utm_medium=member_desktop_web
Let $\mathbf{E}$ is finite dimension linear space with the inner product:

$$(\mathbf{x},\mathbf{y}) \mapsto \langle \mathbf{x},\mathbf{y}\rangle : \mathbf{E}\times\mathbf{E} \to \mathbf{E}.$$

Describe all linear mapping $f : \mathbf{E} \to \mathbf{E}$, which preserve "angles"
(it is mean preserve $\cos(\mathbf{x},\mathbf{y}) = \dfrac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}$).

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1. Let linear mapping $f : \mathbf{E} \to \mathbf{E}$ preserve "angles", then particulary f preserve ortogonality. It is mean:

$$\langle \mathbf{x},\mathbf{y}\rangle = 0 \Leftrightarrow \langle f(\mathbf{x}),f(\mathbf{y})\rangle = 0.$$

For any $\mathbf{x},\mathbf{y} \in \mathbf{E}$, such that $\|\mathbf{x}\|=\|\mathbf{y}\| = 1$ we have $\langle \mathbf{x}-\mathbf{y},\mathbf{x}+\mathbf{y}\rangle = 0.$
($\langle \mathbf{x}-\mathbf{y},\mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{y},\mathbf{x}\rangle - \langle \mathbf{y},\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{y},\mathbf{y}\rangle = 0$).
Hence $\langle f(\mathbf{x}-\mathbf{y}),f(\mathbf{x}+\mathbf{y})\rangle = 0 \Leftrightarrow 0 = \langle f(\mathbf{x})-f(\mathbf{y}),f(\mathbf{x})+f(\mathbf{y})\rangle =$
$\langle f(\mathbf{x}),f(\mathbf{x})\rangle + \langle f(\mathbf{x}),f(\mathbf{y})\rangle - \langle f(\mathbf{y}),f(\mathbf{x})\rangle - \langle f(\mathbf{y}),f(\mathbf{y})\rangle = \|f(\mathbf{x})\|^2 - \|f(\mathbf{y})\|^2.$
So, $\|f(\mathbf{x})\|^2 = \|f(\mathbf{y})\|^2 \Leftrightarrow \|f(\mathbf{x})\| = \|f(\mathbf{y})\|$ i.e.

(1) $\|f(\mathbf{x})\| = const$ for any $\mathbf{x}$ on the unite sphere $S = \{\mathbf{x} \mid \mathbf{x}\in\mathbf{E}$ and $\|\mathbf{x}\| = 1\}$.

Denote this $const$ via d , then, for any $\mathbf{x} \in \mathbf{E}$, $\|f(\mathbf{x})\| = d$,because

$$\left\| f\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\right\| = \left\|\frac{f(\mathbf{x})}{\|\mathbf{x}\|}\right\| = \frac{\|f(\mathbf{x})\|}{\|\mathbf{x}\|} = d.$$

Therefore $\left\langle \mathbf{x},\ (f^*\circ f)(\mathbf{x})\right\rangle = \langle f(\mathbf{x}),f(\mathbf{x})\rangle = \|f(\mathbf{x})\|^2 = d^2\|\mathbf{x}\|^2 = \langle \mathbf{x},d^2\mathbf{x}\rangle \Leftrightarrow$
$\left\langle \mathbf{x},\left(f^*\circ f - d^2e\right)(\mathbf{x})\right\rangle = 0.$
By the other words $\left(f^*\circ f - d^2e\right)(\mathbf{x}) \perp \mathbf{x}$, for any $\mathbf{x} \neq \mathbf{0}$.
Let fix arbitrary $\mathbf{x}\in\mathbf{E}$ and consider representation $\mathbf{E}$ as the direct sum of
$(\mathbf{x}) = Span(\mathbf{x}) = \{k\mathbf{x} \mid k\in\mathbb{R}\}$ and $(\mathbf{x})^\perp$ –ortogonal complement to $(\mathbf{x})$.
So $\mathbf{E} = (\mathbf{x}) \oplus (\mathbf{x})^\perp$ and any $\mathbf{y}\in\mathbf{E}$ can be represented as $\mathbf{y} = k\mathbf{x}+\mathbf{u}$, where $k\in\mathbb{R}$ and $\mathbf{u} \in (\mathbf{x})^\perp$ both depend from $\mathbf{y}$. Then $\left\langle\left(f^*\circ f - d^2e\right)(\mathbf{y}),\mathbf{x}\right\rangle = 0$ for any $\mathbf{y}\in\mathbf{E}$.
Really, $\left\langle\left(f^*\circ f - d^2e\right)(\mathbf{y}),\mathbf{x}\right\rangle = \left\langle\left(f^*\circ f\right)(\mathbf{y}),\mathbf{x}\right\rangle - d^2\langle \mathbf{y},\mathbf{x}\rangle =$
$\langle f(k\mathbf{x}+\mathbf{u}),f(\mathbf{x})\rangle - d^2\langle k\mathbf{x}+\mathbf{u},\mathbf{x}\rangle = k\langle f(\mathbf{x}),f(\mathbf{x})\rangle + \langle f(\mathbf{u}),f(\mathbf{x})\rangle - kd^2\langle \mathbf{x},\mathbf{x}\rangle - d^2\langle \mathbf{u},\mathbf{x}\rangle =$
$k\|f(\mathbf{x})\|^2 - kd^2\|\mathbf{x}\|^2 = 0.$ ($\mathbf{u}\perp\mathbf{x} \Leftrightarrow \langle \mathbf{u},\mathbf{x}\rangle = 0 \Rightarrow \langle f(\mathbf{u}),f(\mathbf{x})\rangle = 0$)
From $\left\langle\left(f^*\circ f - d^2e\right)(\mathbf{y}),\mathbf{x}\right\rangle = 0$ for any $\mathbf{x},\mathbf{y}\in\mathbf{E}$ immediately follows *
$\left(f^*\circ f - d^2e\right)(\mathbf{y}) = 0 \Leftrightarrow (f^*\circ f)(\mathbf{y}) = d^2\mathbf{y}$, for all $\mathbf{y}\in\mathbf{E}$. Thus, $f^*\circ f = d^2e$.
2. Suppose now that $f^*\circ f = d^2e$,
then $\cos(f(\mathbf{x}),\mathbf{f}(\mathbf{y})) = \dfrac{\langle f(\mathbf{x}),f(\mathbf{y})\rangle}{\|f(\mathbf{x})\|\cdot\|f(\mathbf{y})\|} = \dfrac{\langle (f^*\circ f)(\mathbf{x}),\mathbf{y}\rangle}{d^2\|\mathbf{x}\|\cdot\|\mathbf{y}\|} = \dfrac{\langle d^2\mathbf{x},\mathbf{y}\rangle}{d^2\|\mathbf{x}\|\cdot\|\mathbf{y}\|} =$
$\dfrac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|} = \cos(\mathbf{x},\mathbf{y}).$

* Because $\left\langle\left(f^*\circ f - d^2e\right)(\mathbf{y}),\mathbf{x}\right\rangle = 0$ for any $\mathbf{x},\mathbf{y}\in\mathbf{E}$, then, particulary, it is right for $\mathbf{x} = \left(f^*\circ f - d^2e\right)(\mathbf{y})$, so we get $\left\langle\left(f^*\circ f - d^2e\right)(\mathbf{y}),\left(f^*\circ f - d^2e\right)(\mathbf{y})\right\rangle = 0 \Rightarrow$

$$\left(f^*\circ f - d^2e\right)(\mathbf{y}) = 0 \Leftrightarrow f^*\circ f = d^2e.$$